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ON CERTAIN SECOND ORDER EFFECTS IN THE LIMIT DESIGN OF FRAMES

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ON CERTAIN SECOND ORDER EFFECTS IN THE LIMIT DESIGN OF FRAMES[1]

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Abstract: The second order effects due to the changes in geometry are investigated for the frames composed of plastic-rigid material. In the case of ideally-plastic material a criterion is developed to determine whether the quasi-state distortion of the frame proceeds under increasing or decreasing loads. The analysis also shows that when the compressive axial forces are negligible, even a slight amount of strain-hardening considerably increases the load necessary for quasi-static deformations.

1. Introduction.

The plastic methods of analysis and design for beams and frames [1,2] are mainly concerned with the load at which the structure could become a mechanism following the formation of plastic hinges at a sufficient number of sections.

If the changes in geometry caused by the subsequent deformation is neglected, then it can be shown that in the case of frames of ductile material the deformation of the structure proceeds under constant loads. However if this change in geometry is taken into account, then continuing deformation under constant external loads is possible only in exceptional cases [3,4]. Accordingly as a rule, quasi-static flow then requires either increasing or decreasing external loads. If the applied loads are increased monotonically, as is customary, the first case corresponds to a stable process, whereas the second case corresponds to sudden collapse. Since the present first order theory of limit

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analysis does not furnish a criterion which would allow one to distinguish between these two cases, it seems worthwhile to develop a second order theory which should furnish a clear criterion for collapse.

If the frame is composed of a rigid strain-hardening material in which Young's modulus is infinitely large, theorems of limit analysis define a load at which deformation first becomes possible. In this case the load increase to enforce further deformation is governed by the rate of strain-hardening and also by the changes of shape. Because of strain-hardening the plastic deformation in the frame is no longer concentrated at isolated sections and instead this deformation is spread over certain finite segments of the frame.

The purpose of the present paper is to determine the rate at which the loads need to be increased to enforce a quasi-static motion of framed structures made of either ideally-plastic rigid or rigid-plastic strain-hardening material.

It will be shown that in the case of ideally plastic frames an energy criterion can be established to distinguish between the two cases mentioned above. In the case of strain-hardening materials, it is found that if the influence of the axial forces is neglected, then the subsequent deformation can proceed only with increasing load when the rate of strain-hardening is not very small.

As a note of caution it must be added that the second order analysis developed herewith must be supplemented by a study of elasto-plastic stability whenever compressive axial forces of significant intensity exist in the structure.

2. Frames of Ideally-Plastic Rigid Material.

Let us consider a frame composed of ideally-plastic rigid material, and suppose that the loads on this structure are steadily increased in fixed proportion to each other. A critical load is finally reached at which the fully plastic moment M_p is developed a sufficient number of sections, and the structure becomes a mechanism. The calculation of this critical load and the corresponding mechanism is fairly simple in comparison with elastic analyses for highly redundant structures.(2) The subsequent quasi-static motion of this mechanism and the rate of change of the loads to achieve this motion constitute the scope of the present investigation.

In order to fix ideas we shall first consider a very simple model.

The portal frame shown in Fig. 1 is composed of members having the fully plastic moment M_o. It is easily shown that when P reaches the critical value $4M_o/\ell$ plastic hinges develop at the corners A, B, C, D, and the frame becomes a mechanism with a single degree of freedom. The subsequent motion of this mechanism is governed by the basic assumption which states that if at any one cross-section the moment retains the value M_o then this cross-section acts like a hinge (see Fig. 2).

This suggests that the quasi-static deformation of the considered mechanism is only possible if the bending moment remains constant, and equal to M_o, at A, B, C and D.

The resulting four conditions, together with the 3 equilibrium equations yield the following equation for the instantaneous value of P $[= \frac{4M_o}{\ell}(1 + \delta)]$ as a function of θ(see Fig. 1),

$$P = \frac{4M_o}{\ell}(1 + \delta) = \frac{4M_o}{\ell}\frac{\sqrt{2}}{2}\operatorname{cosec}(\theta + \frac{\pi}{4}) \qquad (1)$$

and therefore for small angles of rotation occuring immediately after the yield point has been reached,

$$\delta \approx -\theta.$$

It follows that quasistatic motion can only be maintained with decreasing loads, and so $P = 4M_o/\ell$ is the load at which real collapse occurs.

However if the loads acting had the reverse directions as depicted in Fig. 3, then the same analysis show that the quasi-static flow would require an increase in the magnitude of loading, the instantaneous load now being given by

$$P = \frac{4M_o}{\ell}\frac{\sqrt{2}}{2}\sec(\theta + \frac{\pi}{3}) \qquad (2)$$

Load deformation curves for both cases are shown in Fig. 4. The influence of the axial forces limits the validity of (1) and (2) to the range where θ is small. The above analysis can easily be extended to more complicated cases. However in general this will involve the rather lengthy elimination of internal forces from the equations. In the next section it will now be shown that an alternative and simpler procedure is available.

3. Energy considerations.

We shall now establish an energy criterion for distinguishing between our two collapse cases. In the example of the previous section let us compare the energy dissipation W with the work V done by the original loads after a deformation corresponding to a rotation Θ. If for small rotations

$$W(\Theta) > V(\Theta)$$

we may conclude that quasi-static deformation requires additional work and hence increasing loads. On the other hand if

$$W(\Theta) < V/\Theta)$$

then the loads must decrease. Since

$$\left.\begin{aligned} W(\Theta) &\sim 4M_o\Theta \\ V(\Theta) &\sim 4M_o(\Theta \pm \frac{\Theta^2}{2}) \end{aligned}\right\} \tag{3}$$

the positive sign being taken in the first case (Fig. 1) and the negative sign in the second case Fig. (3).

$$V(\Theta) - W(\Theta) \sim \pm 2M_o\Theta^2 \tag{4}$$

which is in agreement with the conclusions based upon the statical considerations mentioned earlier. This procedure can be generalized without difficulty to more complicated cases.

4. Frames of Rigid Strain-Hardening Materials.

For the purpose of the present investigation, it is sufficient to consider a frame composed of material having the idealized uni-axial stress-strain diagram shown in Fig. 5. For continuous loading in tension or compression, we have then the following stress-strain relation;

$$|\sigma| = E_p|\varepsilon| + \sigma_o \tag{5}$$

where all symbols are defined in Fig. 5.

If the cross-section of the beam is doubly-symmetric and if the bending moment lies along an axis of symmetry, then the classical assumption that the cross-sections remain plane during bending yields the following relationship between the bending moment M and the radius of curvature ρ of the neutral axis:

$$\left.\begin{aligned} \frac{M-M_o}{E_p I} &= \frac{1}{\rho} \qquad \text{when} \quad |M| > |M_o|, \\ \frac{1}{\rho} &= 0 \qquad \text{when} \quad |M| \leq |M_o|, \end{aligned}\right\} \tag{6}$$

where E_p is the plastic tangent modulus of the material, F is the cross-sectional moment of inertia and M_o is the initial yield moment.

Now, let us investigate the state of affairs in the neighbourhood of a yield hinge H following the incipient motion [Fig. 6]. Suppose that during the subsequent motion the bending moment at the cross-sections in the neighborhood of H increases.

According to (6) the neutral axis of the beam will be curved in the region AB, and this curved part of the beam will generally change its size and shape during the deformation.

Since at the end points of the curve AB

$$\frac{1}{\rho} = 0,$$

the bending moment computed on the straight line extension AB' and H'B of the members 1 and 2 can reasonably be expected to be sufficiently close to the actual bending moment. This suggests that the initial deformation of the frame is closely approximated by the deformation of a frame having ideal yield hinges. However, the simple yield condition for ideally plastic frames should be replaced by an equation expressing the relationship between the angle of relative rotation Θ_{AB} of two neighboring rigid portions and the bending moment distribution.

From (6)

$$\Theta_{AB} = \int_A^B \frac{ds}{\rho} = \int_A^B \frac{M-M_o}{E_p I}\, ds,$$

and therefore since the bending moments are computed along straight lines AH' and H'B, it follows that

$$E_p I \Theta_{AB} = \frac{1}{V_1}\int_{M_o}^{M_{max}} (M-M_o)\,dM + \frac{1}{V_2}\int_{M_o}^{M_{max}} (M-M_o)\,dM,$$

or

$$E_p I \Theta_{AB} = \left(\frac{1}{V_1} + \frac{1}{V_2}\right)(M_{max}-M_o)^2, \qquad (7)$$

where V_1 and V_2 are the absolute values of the shear forces [Fig. 6].

Eq. (7) is the condition for strain hardening which replaces $M_{max} = M_o$ for perfectly plastic frames. After these preliminary remarks on the incipient bending deformations of rigid strain-hardening beams, we may now investigate second order effects, viz. the determination of the rate at which the loads need to be increased to enforce quasi-static motions . To illustrate the preceding theory let us consider the simple example shown in Fig. (7). The yield point is reached when $P = P_o$ where

$$P_o = \frac{M_o}{\ell \cos \alpha} \qquad (8)$$

and H becomes a yield hinge. If the angle of rotation of one of the bars is denoted by θ, the condition (7) applied at H gives

$$2E_pI\theta = \frac{[P\ell\cos(\alpha-\theta) - M_o]^2}{P\cos(\alpha-\theta)}$$

or

$$\frac{P}{P_o} = \frac{\cos\alpha}{\cos(\alpha-\theta)}\left[1 + \varepsilon\theta + \left\{\varepsilon\theta(2+\varepsilon\theta)\right\}^{1/2}\right] \qquad (9)$$

where

$$\frac{E_pI}{M_o\ell} = \varepsilon .$$

The corresponding load deformation curves are shown in Fig. 7 for a typical shape of frame and for various degrees of strain-hardening. It is clear that even a slight amount of strain-hardening will considerably increase the load necessary for quasi-static deformations of the structure.

For example, if the frame cross-section is rectangular

$$\varepsilon = \frac{1}{3}\frac{E}{\sigma_o}\frac{h}{\ell}$$

where h is the depth of the bar. For the relatively small degree of strain-hardening $\frac{E_p}{\sigma_o} = 1.5$, and for $\frac{\ell}{h} = 10$; $\varepsilon = .05$.

Whereas in the case of ideally plastic material $P = P_o$ is a real collapse load, a slight amount of strain hardening of the material necessitates increasing loads with increasing deformations (Fig. 7).

References.

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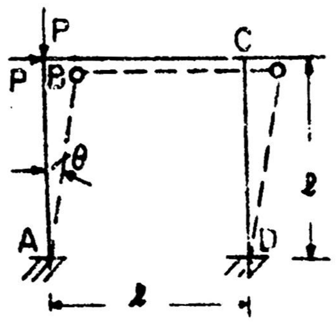

Fig. 1

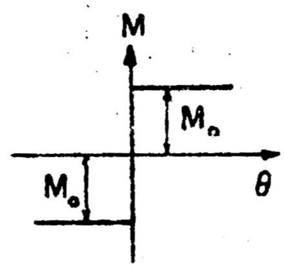

Fig. 2

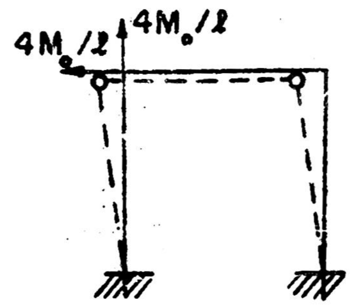

Fig. 3

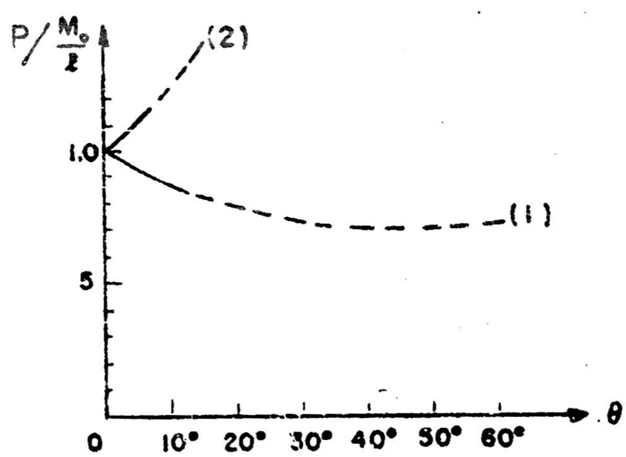

Fig. 4

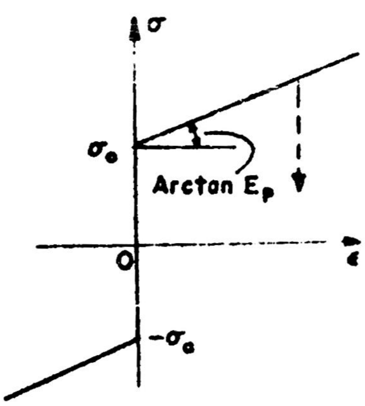

Fig. 5

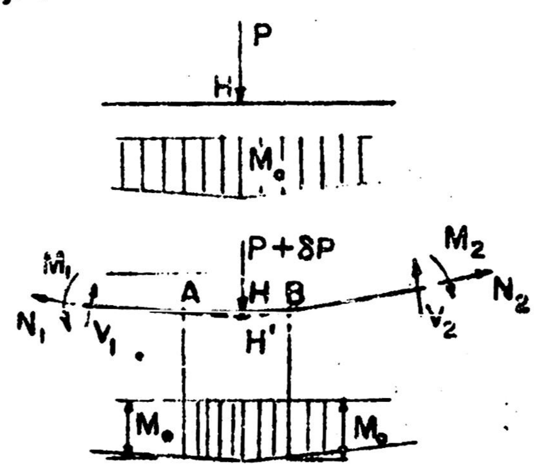

Fig. 6

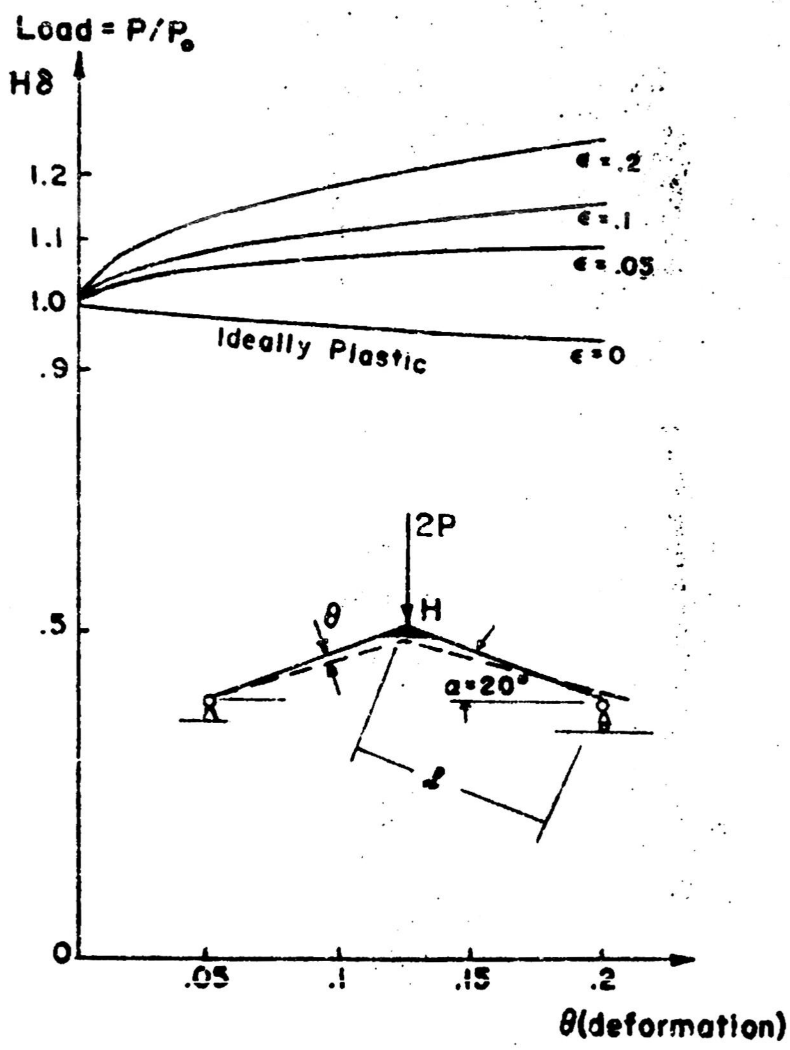
Load = P/P_o
H8
1.2
1.1
1.0
.9
.5
0
$\epsilon = .2$
$\epsilon = .1$
$\epsilon = .05$
$\epsilon = 0$
Ideally Plastic
2P
H
θ
$\alpha = 20°$
ℓ
.05
.1
.15
.2
θ(deformation)

Fig. 7